Journal of Engineering Physics and Thermophysics, Vol. 71, No. 6, 1998

SLOW FLOWS BETWEEN COAXIAL CONICAL SURFACES

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UDC 532.58; 678.027

The problems of a slow Newtonian-fluid flow in channels formed by coaxial cones with or without a common apex are solved. Convenient relations for calculating the drop in pressure and velocity are obtained.

Recently the technical significance of highly viscous fluids, in particular, in treatment and production of plastics and items made of them, has increased considerably. Flows in apparatuses for treatment of polymers take place in channels of different shapes. In the majority of the designs of extrusion press molds and die, cable, and tube heads [1, 2] there exists a portion where flows occur between conical surfaces (Fig. 1).

To choose optimum technological and design parameters of extrusion processes one should develop reliable, scientifically grounded methods for calculation of the flow parameters in channels of extrusion heads, whose pressure-flow rate characteristic determines the working point of the extruder [1].

Within the limits of variation of the treatment parameters, melts of some polymers behave like a Newtonian fluid [3]. For practically important flow rates of these fluids $Q \approx 0.5 \cdot 10^{-4}$ m^3/sec, the rheophysical properties $\mu \sim 10^3$ Pa·sec, $\rho \sim 1250$ kg/m^3, $\lambda \sim 0.2$ W/(m·K), $\Delta T_{\mathrm{rheol}} \sim 6$ K [4], and the geometric dimensions of the channels (Fig. 1) the Nem–Griffith number Gn << 1 [4, 5], and the Reynolds number Re << 1.

The value of the Gn number indicates that dissipative effects do not affect the flow dynamics and they can be neglected, which together with good thermostating of extruders [2] makes it possible to consider the flow in annular conical channels as isothermal.

A laminar flow between coaxial cones with a common apex was considered in [6] in spherical coordinates. An analytical solution is obtained for the general case with account for inertia terms in the equations of motion. The form of this solution makes its use in practical calculations difficult. In [7] a general solution for spherical flows is obtained irrespective of the boundary conditions, and in [8] a flow of a nonlinear-viscous fluid between cones with a constant width of the gap between them is studied by the method of finite elements. The authors of [8] determine the velocity field and the flow rate from the given pressure drop, although in engineering practice the solution of the inverse problem is necessary, as a rule. In [2, 9] it is suggested that conical flows be calculated by stepwise approximation by cylindrical channels, which can lead to considerable errors.

The above estimates allow one to obtain simple expressions for calculation of flows between round conical surfaces within a wide range of variation of the geometric parameters. These expressions take into account the shape of the channel.

First, we consider a flow between coaxial cones with a common apex. A small Reynolds number allows one to simplify the equation of motion and, following [4], to write them with account for the axial symmetry of the flow in the form

$$\frac{1}{\mu}\frac{\partial P}{\partial R} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V_R}{\partial R}\right) + \frac{1}{R^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial V_R}{\partial \theta}\right) - \frac{2V_R}{R^2}, \tag{1}$$

$$\frac{1}{\mu}\frac{\partial P}{\partial \theta} = \frac{2}{R}\frac{\partial V_R}{\partial \theta}, \tag{2}$$

Kharkov State Polytechnic University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 71, No. 6, pp. 1092-1098, November-December, 1998. Original article submitted March 4, 1997.

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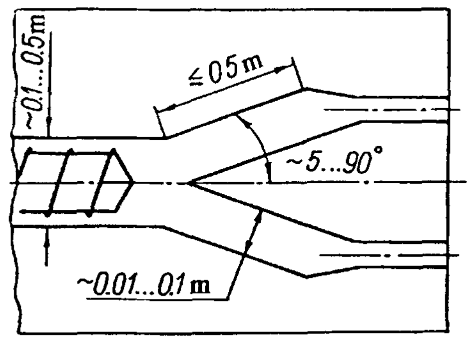

Fig. 1. Cross section of a typical die head.

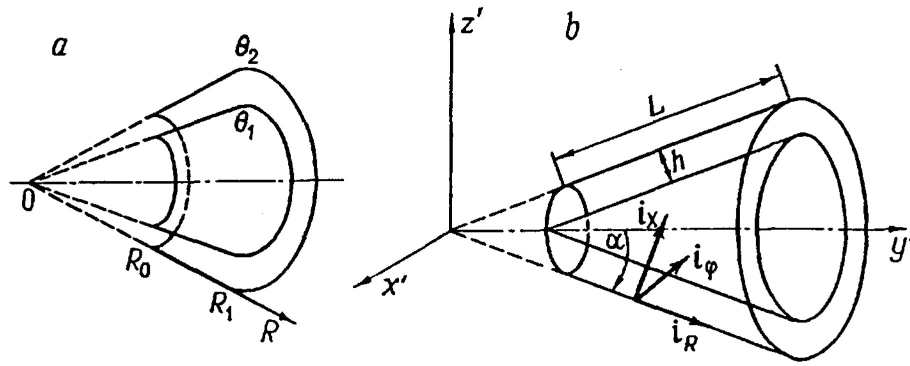

Fig. 2. Geometry of a conical gap: a) cones with a common apex (R, R_0, R_1, radial coordinate, coordinates at the inlet and outlet, m; θ_1, θ_2, apex angles of cones, deg); b) cones without a common apex (L, length of the conical portion of the channel, m; h, gap width, m; $\mathbf{i}_R$, $\mathbf{i}_X$, $\mathbf{i}_\varphi$, unit vectors in a biconical system of coordinates).

$$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 V_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta) = 0\,, \tag{3}$$

with the boundary conditions for coaxial cones with a common apex (Fig. 2)

$$V_R = 0\,,\ \ V_\theta = 0\,,\ \ \theta = \theta_1\,, \tag{4}$$

$$V_R = 0\,,\ \ \theta = \theta_2\,, \tag{5}$$

$$\bar{P} = 0\,,\ \ R = R_1\,, \tag{6}$$

where

$$\bar{P}(R) = \frac{1}{\cos\theta_1 - \cos\theta_2}\int_{\theta_1}^{\theta_2} P(R,\theta)\sin\theta d\theta\,.$$

For aperture angles of the cones $\Delta\theta = \theta_2 - \theta_1 \le 30^\circ$ the continuity equation (3) makes it possible to estimate the angular component of the velocity $V_\theta \sim \Delta\theta V_R$, i.e., $V_\theta = o(V_R)$, and to neglect it in the equations of motion and continuity, which in the dimensionless variables

$$\xi = \frac{R}{r_0}, \quad v = \frac{V_R}{V_0}, \quad \Pi = \frac{(P - P_0)\, r_0}{\mu V_0}, \quad \tau = \cos\theta, \quad V_0 = \frac{Q}{2\pi R_0^2 (\tau_1 - \tau_2)}, \tag{7}$$

are written as

$$\frac{\partial \Pi}{\partial \xi} = \frac{1}{\xi^2} \frac{1}{\partial \xi} \left(\xi^2 \frac{\partial v}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \tau} \left[(1 - \tau^2) \frac{\partial v}{\partial \tau} \right] - \frac{2v}{\xi^2}, \tag{8}$$

$$\frac{\partial \Pi}{\partial \tau} = \frac{2}{\xi} \frac{\partial v}{\partial \tau}, \tag{9}$$

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} (\xi^2 v) = 0. \tag{10}$$

The boundary conditions take the form

$$v = 0, \quad \tau = \tau_1; \tag{11}$$

$$v = 0, \quad \tau = \tau_2; \tag{12}$$

$$\overline{\Pi} = 0, \quad \xi = \xi_0, \tag{13}$$

and

$$\overline{\Pi} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Pi d\tau.$$

The condition of flow-rate constancy is written as

$$\int_{\tau_1}^{\tau_2} v d\tau = - \xi_1^2 (\tau_1 - \tau_2)/\xi^2. \tag{14}$$

The dependence $v = u(\tau)/\xi^2$ follows from (10), which in combination with (9) gives

$$\frac{\partial \Pi}{\partial \xi} = - \frac{6}{\xi^4} u + f'(\xi), \tag{15}$$

Substituting (15) in (7) and separating the variables, we obtain equations for determining $u(\tau)$ and $f(\xi)$:

$$(1 - \tau^2) \frac{\partial^2 u}{\partial \tau^2} - 2\tau \frac{\partial u}{\partial \tau} + 6u = - \lambda, \tag{16}$$

$$\xi^4 f'(\xi) = - \lambda. \tag{17}$$

The solution of (16) with account for conditions (11)-(14) has the form

$$u = \frac{\lambda}{6} (AP_2(\tau) + BQ_2(\tau) - 1), \tag{18}$$

where

$$A = \frac{Q_2(\tau_2) - Q_2(\tau_1)}{P_2(\tau_1) Q_2(\tau_2) - P_2(\tau_2) Q_2(\tau_1)}; \quad B = \frac{P_2(\tau_2) - P_2(\tau_1)}{P_2(\tau_1) Q_2(\tau_2) - P_2(\tau_2) Q_2(\tau_1)};$$

$$\lambda = \frac{6\xi_1^2}{C + \frac{B}{4}\left[\frac{\tau_2(\tau_2^2-1)}{\tau_1-\tau_2}\ln\frac{1+\tau_2}{1-\tau_2} - \frac{\tau_1(\tau_1^2-1)}{\tau_2-\tau_1}\ln\frac{1+\tau_1}{1-\tau_1} - 2(\tau_2+\tau_1)\right] - 1},$$

$$C = \frac{A}{2}(\tau_2^2 + \tau_2\tau_1 + \tau_1^2 - 1),$$

and $P_2(\tau) = 0.5(3\tau^2 - 1)$, $Q_2(\tau) = (1/2)P_2(\tau)\ln[(1+\tau)/(1-\tau)] - (3/2)\tau$ are Legendre polynomials of the first and second kind and second order.

Using (12), (14), (16), and (17), we write an expression for the pressure distribution:

$$\Pi(\xi, \tau) = \frac{2u(\tau)}{\xi^3} + \frac{\lambda}{3}\left(\frac{1}{\xi^3} - \frac{1}{\xi_0^3}\right) - \frac{2}{\xi_0}, \tag{19}$$

and, averaging it over the channel cross section, we have

$$\overline{\Pi}(\xi) = \frac{\lambda + 6\xi_0^2}{3}\left(\frac{1}{\xi^3} - \frac{1}{\xi_0^3}\right). \tag{20}$$

Setting $\theta_2 = \pi/2$, we find the solution for a slow axisymmetric radial flow between a plane and a cone with the apex lying on the plane and the axis perpendicular to it.

The design of the portion of the distribution of a melt flow formed by coaxial cones with a common apex is a particular case of the design of extrusion heads. Annular conical channels can be formed by concentric surfaces with a constant distance between them or with an arbitrary law of gap width variation. Therefore, to select optimum design and technological parameters we should study the flow between coaxial cones without a common apex.

We consider an annular conical channel of constant width (Fig. 1). The fluid parameters and the basic geometric dimensions of devices for moulding are similar to those given above. Therefore, the fluid flow can be considered isothermal, and inertia terms in the equations of motion can be neglected. It is convenient to study the equations of hydrodynamics in this channel in biconical coordinates [10] whose origin coincides with the apex of the outer conical surface (Fig. 1), defined by the transformations

$$z' = R\cos\alpha + X\sin\alpha, \tag{21}$$

$$y' = (R\sin\alpha - X\cos\alpha)\sin\varphi = \Omega\sin\varphi, \tag{22}$$

$$x' = (R\sin\alpha - X\cos\alpha)\cos\varphi = \Omega\cos\varphi. \tag{23}$$

Calculating the Lamé coefficients $H_X = 1$, $H_R = 1$, $H_\varphi = \Omega$ and following [11], we write the equations of continuity and motion in the selected coordinates.

The continuity equation for an axisymmetric flow is

$$\frac{\partial}{\partial R}(\Omega V_R) + \frac{\partial}{\partial X}(\Omega V_X) = 0. \tag{24}$$

Hence we obtain an estimate of the relation between the quantities V_X and V_R, $V_X \approx 2V_R h/L$, i.e., $V_X = o(V_R)$. This allows the system of equations of hydrodynamics in the dimensionless variables

$$\xi = R/r_0\,,\ \ \chi = X/r_0\,,\ \ V_0 = Q/(\pi r_0^2)\,,\ \ v = V_R/V_0\,,\ \ \Pi = (P - P_0)\, r_0/\mu V_0 \tag{25}$$

to be written as

$$\frac{\partial \Pi}{\partial \xi} = \frac{1}{\sigma}\frac{\partial}{\partial \chi}\left(\sigma \frac{\partial v}{\partial \chi}\right), \tag{26}$$

$$\frac{\partial \Pi}{\partial \chi} = \frac{\cos(\alpha)\sin(\alpha)}{\sigma^2}\, v\,, \tag{27}$$

$$\frac{\partial}{\partial \xi}(\sigma v) = 0\,, \tag{28}$$

where $\sigma = \xi \sin\alpha - \chi \cos\alpha$.

In cases of practical interest the condition $\xi \tan\alpha >> \chi$ is always fulfilled, i.e., we can set $\sigma \approx \xi \sin\alpha$. Having estimated the derivaties in the system of equations, we obtain $\partial\Pi/\partial\xi \approx v/\chi_0$, $\partial\Pi/\partial\chi \approx v/\xi \tan\alpha$, i.e., we can set $\partial\Pi/\partial\xi = 0$ and reduce the system of equations (26), (27) to the following:

$$\frac{\partial \Pi}{\partial \xi} = \frac{\partial^2 v}{\partial \chi^2}. \tag{29}$$

The boundary conditions are the condition of sticking and the assigned pressure at the channel inlet:

$$u = 0\,,\ \ \chi = 0\,; \tag{30}$$

$$u = 0\,,\ \ \chi = \chi_0\,,\ \ (\chi_0 = h/r_0)\,; \tag{31}$$

$$\Pi = 0\,,\ \ \xi = \xi_0\,, \tag{32}$$

and the continuity equation is accounted for by the condition of flow-rate constancy

$$\int_0^{\chi_0} (\xi \sin\alpha - \chi\cos\alpha)\, v d\chi = 0.5\,. \tag{33}$$

The solution of the system (29)-(32) is the expressions

$$v = \frac{6\,(\chi^2 - \chi_0\chi)}{\chi_0^3\,(\chi_0\cos\alpha - 2\xi\sin\alpha)}\,, \tag{34}$$

$$\Pi = -\frac{6}{\chi_0^3 \sin\alpha}\ln\frac{\chi_0 - 2\xi\tan\alpha}{\chi_0 - 2\xi_0\tan\alpha}\,. \tag{35}$$

We note two limiting cases ensuing from (35):

1) when $\alpha \to 90^\circ$, we obtain an expression for determining the pressure drop in a radial flow between parallel planes:

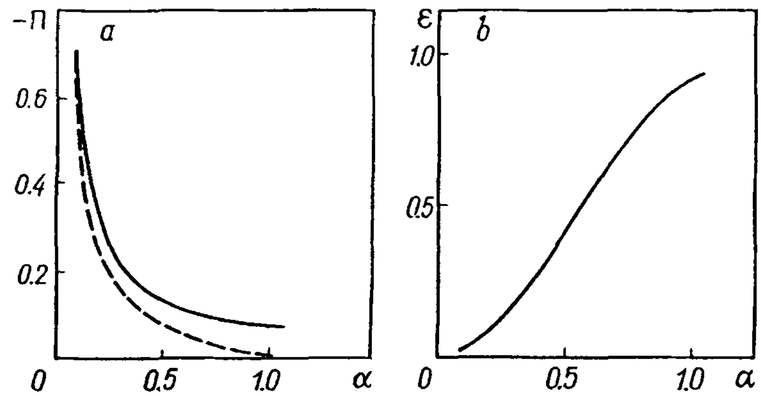

Fig. 3. Dependences of (a) the dimensionless pressure drop in the channel (solid line – calculation by (35), dashed line – by (40) and (40a) at $N = 60$) and (b) the relative error for the aforementioned quantities on the apex angle of the cone for a conical annular channel with a constant gap width.

$$\Pi = -\frac{6}{\chi_0^3} \ln \frac{\xi}{\xi_0}, \tag{36}$$

2) when $\chi_0/\xi_0 \ll 1$, we have an expression for the pressure drop in narrow conical slits [12]:

$$\Pi = -\frac{6}{\chi_0^3 \sin\alpha} \ln \frac{\xi}{\xi_0}, \tag{37}$$

which can lead to large errors in cases where χ_0 is comparable to the radius of curvature of the channel.

In calculations of the pressure drop in coaxial conical channels whose gap width is not constant along the flow, e.g., the generatrix of the inner surface is defined as $\chi_0 = f(\xi)$, it is possible to use a stepwise approximation:

$$\Pi = -\frac{6}{\sin\alpha} \sum_{i=1}^{N} \frac{1}{\chi_{0i}^3} \ln \frac{\chi_{0i} - 2\xi_{i+1}\tan\alpha}{\chi_{0i} - 2\xi_i \tan\alpha}, \tag{38}$$

where $\xi_{i+1} = \xi_0 + (\xi_N/N)(i-1)$; $\chi_{0i} = f(\xi_i)$; N is the number of steps; ξ_0 is the inlet coordinate; ξ_N is the outlet coordinate. If the function $f(\xi)$ is unknown, an approximate representation of it should be used.

Similarly an expression for calculating the pressure in an axisymmetric annular channel of variable cross section with an arbitrary shape of the channel generatrix is written:

$$\Pi = -6 \sum_{i=1}^{N} \frac{1}{\chi_{0i}^3 \sin\alpha_i} \ln \frac{\chi_{0i} - 2\xi_{i+1}\tan\alpha_i}{\chi_{0i} - 2\xi_i \tan\alpha_i}, \tag{39}$$

where all quantities are determined for each portion of the channel.

We note that this approximation allows for the geometric features of the channel and does not replace a converging or diverging conical flow by a straight flow.

Moreover, if in (38) we set $\alpha = 90^\circ$, then, in combination with (34), we obtain approximate expressions for calculation of a slow radial flow between a plane and a cone with an arbitrarily located apex.

In [2, 9] the pressure drop in an annular conical channel is suggested to be calculated by a stepwise approximation of the channel by coaxial cylinders. Having applied this approach to a channel of constant width, following [2] we have the relation

$$\Delta\Pi = -\frac{24(\xi_1 - \xi_0)\cos^5\alpha}{\chi_0^3[(\xi_1 + \xi_0)\sin 2\alpha - 2\chi_0]} \tag{40}$$

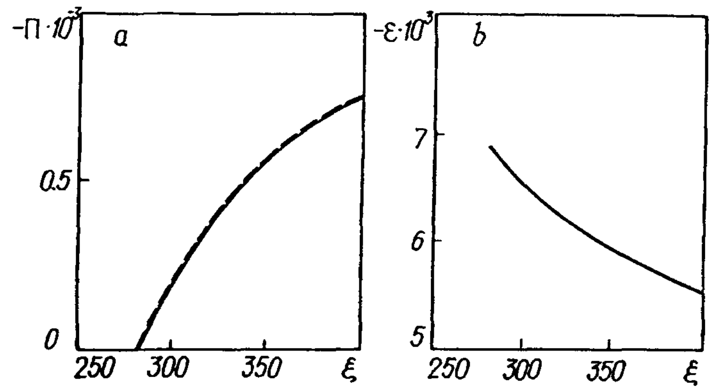

Fig. 4. Dependences of (a) the dimensionless pressure drop in the channel (solid line – calculation by (20), dashed linc – by (38) at $N = 60$) and (b) the relative error for the aforementioned quantities on the dimensionless radius for a conical annular channel formed by cones with a common apex.

or, following [9], the expression

$$\Delta\Pi = -12\frac{\xi_1 - \xi_0}{N\chi_0}(\cos\alpha)^4 \sum_{i=1}^{N} \frac{1}{2\xi_i \sin\alpha - \chi_0 \tan\alpha}, \tag{40a}$$

which give virtually the same result for $N > 20$.

A comparison of the results of calculation of the pressure drop in a conical channel with a constant gap width at $\chi_0 = 3.33$, $\xi_0 = 280$, $\xi_1 = 400$ by (35) and (40), (40a) shows a large difference between them, especially at apex angles of the cones $\alpha > 30^\circ$ (Fig. 3). The application of a stepwise approximation to the flow between cones with a common apex gives even greater divergence from the results obtained by (20).

The results obtained by (38) in this case show good convergence with relation (20), which in the variables (25) is written as

$$\overline{\Pi} = \frac{\lambda + 6\xi_0^2}{6\xi_0^2(\tau_1 - \tau_2)}\left(\frac{1}{\xi^3} - \frac{1}{\xi_0^3}\right), \tag{41}$$

for $\theta_1 = 10.91^\circ$, $\theta_2 = 15^\circ$, $\xi_0 = 280$, $\xi_1 = 400$ (Fig. 4) and also within the entire range of variation of the parameters.

The results presented here along with works performed by the author earlier [4, 5, 13, 14] allow selection of optimum technological and design parameters of extrusion heads and determination of their pressure-flow rate characteristic, using which one can determine the working point of the extruder.

NOTATION

d, equivalent diameter, m; V, velocity; T, temperature; P, P_0, current pressure and pressure at the inlet, Pa; Q, volumetric flow rate, m^3/sec; R, R_0, R_1, radial coordinate and coordinates at the inlet and outlet of the channel, m; r_0, nondimensionalization parameter, as a rule, the radius of the die channel, m; x', y', z', Cartesian coordinates, m; α, apex half-angle of the cone, rad; λ, separation constant; μ, viscosity, Pa·sec; θ, angular coordinate, rad; X, transverse biconical coordinate, m; $\mathrm{Gn} = \mu V^2/\lambda\Delta T_{\mathrm{rheol}}$, Nem–Griffith number; $\mathrm{Re} = dV\rho/\mu$, Reynolds number.

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